

# Zerofree region for exponenetical sums

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§1 We consider the following two closed sets in  $C^n$ . One is the diagonal  $D$  given by  $(z, z, z, \dots, z)$ . The other is  $A = \{(z_1, z_2, z_3, \dots, z_n) : e^{z_1} + e^{z_2} + e^{z_3} + \dots + e^{z_n} = 0\}$ . Clearly  $D \cap A$  is empty. One can ask what is the distance between them.

In this connection, Stolarsky [1] proved that the distance  $d$  is given by  $d^2 = (\log n)^2 + O(1)$ . Some simple calculations will make one believe that the point  $(k, 0, 0, \dots, 0)$  with  $k = \log(n-1) + \pi i$  which lies on  $A$  is one of the closest point to the diagaonal. We prove that this is indeed the case, atleast for sufficiently large  $n$ . This gives  $d^2 = |k|^2(1 - 1/n)$ .

§2. We first make the following observations.

If  $P(z_1, z_2, \dots, z_n)$  is any point in  $C^n$ , then the nearest point in  $D$  to  $P$  is  $(z, z, z, \dots, z)$  where  $z = (z_1 + z_2 + z_3 + \dots + z_n)/n$ . Thus if  $P(z_1, z_2, z_3, \dots, z_n)$  is a point of  $A$  closest to  $D$ , so are the points  $(\bar{z}_1, \bar{z}_2, \dots, \bar{z}_n)$ ,  $(z_1 - a, z_2 - a, z_3 - a, \dots, z_n - a)$  and  $(z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)}, \dots, z_{\sigma(n)})$  for any  $a$  in  $C$  and for any permutation  $\sigma$  of  $(1, 2, 3, \dots, n)$ .

Thus one may assume that

$$e^{z_1} + e^{z_2} + e^{z_3} + \dots + e^{z_n} = 0 \quad (1)$$

$$\text{Im}(z_n) \geq 0 \quad (2)$$

$$\sum_{j=1}^n z_j = 0 \quad (3)$$

$$|z_1| \leq |z_2| \leq \dots \leq |z_n| \quad (4)$$

Further we can assume that

$$\text{Im} z_n \leq \pi \quad (5)$$

since otherwise the point  $(z_1, z_2, \dots, z_{n-1}, z_n - 2\pi i)$  is nearer to  $D$

Now consider the point  $Q(b_1, b_2, b_3, \dots, b_n)$  where  $b_j = -k/n$  for  $1 \leq j \leq (n-1)$  and

$$b_n = k - k/n$$

Then  $Q$  satisfies the conditions above and is at a distance  $|k|^2(1 - 1/n)$  from  $D$ .

Our proof will be complete if we prove that there is no point at a smaller distance.

Thus we assume (1), (2), (3), (4), (5) and

$$|z_1|^2 + |z_2|^2 + |z_3|^2 + \dots + |z_n|^2 \leq |k|^2(1 - 1/n) \quad (6)$$

$$\text{Define } a_j = z_j - b_j \quad (7)$$

our aim is to prove that  $a_j = 0$  for all  $j$

$$\text{We note that } \sum_{j=1}^n a_j = 0 \quad (8)$$

**Lemma 1** We have  $\sum_{j=1}^n |a_j|^2 = O(\log n |a_n|)$

**Proof** We have, from equations (6) and (7)

$$\sum_{j=1}^n |a_j + b_j|^2 \leq \sum_{j=1}^n |b_j|^2$$

$$\text{Hence } \sum |a_j|^2 \leq -2Rl \sum (\bar{b}_j a_j) = 2|\sum b_j a_j|$$

Substituting the value of  $b_j$  and using (8) we get

$$\sum |a_j|^2 \leq 2|k||a_n| \text{ and hence the lemma.}$$

**Lemma 2** We have  $\sum_{j=1}^{n-1} (e^{a_j} - 1) = (n-1)(e^{a_n} - 1)$

**Proof** We have  $0 = \sum_{j=0}^n e^{z_j} = \sum e^{(b_j + a_j)}$

Now we substitute the value of  $b_j$  to get  $\sum_{j=1}^{n-1} e^{a_j} = (n-1)e^{a_n}$ . The lemma follows.

**Lemma 3** We have  $\sum_{j=1}^{n-1} (e^{a_j} - a_j - 1) = (n-1)(e^{a_n} - 1) + a_n$

**Proof** This follows from lemma2 and (8)

**Lemma 4** We have  $\sum |a_j|^2 = O(\log^2 n)$

**Proof** Since  $|a_j| \leq |z_j| + |b_j|$ , this follows from equation (6) and definition of  $b_j$ .

Now define  $M = \max |a_j|$ , the maximum over  $1 \leq j \leq n-1$ .

**Lemma 5** We have,  $M \leq 0.75 \log n$

**Proof** We have, if  $j \leq (n-1)$ , then  $2|z_j|^2 \leq \sum_{j=1}^n |z_j|^2 \leq (\log n)^2 + \pi^2$  by equation (4) and (6). Hence  $|z_j| \leq 0.72 \log n$ .

Since  $|a_j| \leq |z_j| + |b_j|$ , the lemma follows

**Lemma 6** We have  $\sum_{j=1}^{(n-1)} (e^{a_j} - 1 - a_j) = O(n^{0.75})$

**Proof** Let  $S_r = a_1^r + a_2^r + \dots + a_{(n-1)}^r$ .

Then  $S_1 = -a_n$  and for  $l \geq 2$ ,  $|S_l| \leq S_2 M^{(l-2)}$  where  $M$  is the maximum of  $a_j$  which is  $0.75 \log n$  in our case.

Now  $e^{a_j} - 1 - a_j = \sum_{l=2}^{\infty} a_j^l / l!$ .

$$\begin{aligned} &\text{Summing over } j = 1 \text{ to } (n-1) \\ &\sum_{j=1}^{(n-1)} (e^{a_j} - 1 - a_j) = \sum_{l=2}^{\infty} S_l / l! \end{aligned}$$

Now using the bound on  $M$  and noting that  $S_2$  is  $O(\log n)^2$  by Lemma 4, the result follows.

**Lemma 8.** We have  $a_n = O(n^{-0.25})$

**Proof** By Lemmas 3 and 6, we have  $(n-1)|e^{a_n} - 1| = O(n^{0.75}) + O(|a_n|)$ .

Since  $a_n = O(\log n)$  by Lemma 4, we get  $e^{a_n} - 1$  is small. This means,  $a_n$  is small, upto a multiple of  $2\pi i$ . Now the assumption (2) and (5) ensure that the multiple is zero.

Now Lemma 8, combined with Lemma 1 shows that all  $a_j$  are small. With this information, we relook at Lemma 3. Now the right side of Lemma 3 is  $\gg n|a_n|$  and the left side is  $\ll \sum_{j=1}^{n-1} |a_j|^2$  which is  $O(\log n|a_n|)$  by Lemma 1.

Thus  $n|a_n| \ll \log n|a_n|$ . This easily leads to a contradiction unless  $a_n = 0$ . Then by Lemma 1, all  $a_j$  are zero. Hence the result.

Reference:

[1]: Stolarsky; K.B: Zero free regions of exponential sums. Proc. A.M.S.89 (1987) (486-488).